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Towards the Full Lyapunov Spectrum of Elementary Cellular Automata

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Abstract. Throughout the decades following the postulation of cellular automata (CA) halfway the previous century, numerous studies have been conducted to gain insight into the dynamical properties of these uttermost discrete dynamical systems. Mostly, these studies were motivated by the fact that CA turned out capable of evolving intriguing spatio-temporal dynamics notwithstanding their intrinsically simple nature. Though many measures have been proposed for gaining insight into CA dynamics, the use of Lyapunov exponents that measure how their phase space trajectories evolve with respect to each other has proved particularly fruitful. Yet, to this day, in all studies relying on this measure the conclusions are drawn upon the so-called maximum Lyapunov exponent (MLE), whereas the determination of the full Lyapunov spectrum has been neglected despite its demonstrated usefulness in elucidating the dynamics of continuous dynamical systems and maps. In this preliminary study, we outline how the Lyapunov spectrum of so-called elementary CA can be obtained, we show the validity of the proposed computational methodology and, finally, we present the full spectra of some renowned elementary CA.

Keywords: Cellular automata, Lyapunov spectrum, Stability
PACS: 05.45

INTRODUCTION

Though models based upon the CA paradigm, as conceptualized by von Neumann halfway the previous century [1, 2], are becoming increasingly popular as illustrated by their appearance in various scientific disciplines [e.g. 3, 4, 5, 6, 7], CA remain largely renowned for the intriguing spatio-temporal dynamics they can bring forth in spite of their simple nature. Therefore, many researchers in the field of nonlinear dynamics established methods and methodologies that give insight into CA dynamics, as such bypassing the need for a qualitative assessment of the evolved space-time patterns by means of visual inspection. These studies have resulted in a panoply of dynamics-grasping measures, such as the Langton parameter [8], the Hamming distance [9, 10], compression-based quantifiers [11], entropies and dimensions [12], and others [13]. Still, the most often resorted to is probably the maximum Lyapunov exponent (MLE), of which a non-directional [14] as well as a directional [10, 15] variant have been postulated. In contrast to its directional counterpart, the non-directional MLE has a much wider applicability, since it is not confined to one-dimensional CA [15], and, its upper bound can be determined straightforwardly by relying on a statistical measure that expresses the sensitivity of a cellular automaton to its inputs [10, 15], and which can be computed readily by relying on the notion of Boolean derivatives [16]. Although the conclusions in all of the aforementioned studies are drawn upon the MLE, it should not be forgotten that a cellular automaton is a discrete dynamical system of which the phase space's dimensionality agrees with its number of spatial entities, commonly referred to as cells, such that it possesses a full Lyapunov spectrum composed of as many exponents as the number of cells on which the cellular automaton is based. Therein, the MLE is merely the largest Lyapunov exponent and has been used so far to gain insight into CA dynamics, though CA have been contemplated as higher-dimensional dynamical systems in previous studies [e.g. 17, 18, 19]. Yet, motivated by the important role the full Lyapunov spectrum plays in the characterization of dynamical systems that are based upon a continuous state domain [20, 21], we show in this paper how this spectrum can be computed and indicate the validity of the presented computational methodology. An outline of the procedure for assessing the Lyapunov spectrum is presented in the first section of this paper, while its validity is checked in this paper's final section .

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LYAPUNOV EXPONENTS OF CELLULAR AUTOMATA

Before turning to the procedure that should be followed for computing the Lyapunov spectrum of CA, we briefly review the state-of-the-art with regard to the MLE of CA.

The maximum Lyapunov exponent

According to Bagnoli et al. [10], the MLE (λ_1) of a one-dimensional cellular automaton, with state space $S = \{0,1\}$ and n spatial entities c_i that cover $\mathbb{R}$, *i.e.* a so-called elementary cellular automaton, is given by:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{\varepsilon_t}{\varepsilon_0}\right), \tag{1}$$

where ε_t represents the number of cells c_j of which the state is perturbed at the t-th time step during the cellular automaton's evolution following the introduction of a single perturbation at $t = 0$ ($\varepsilon_0 = 1$), *i.e.* the number of defects. As such, λ_1 quantifies the rate of divergence/convergence between trajectories in phase space that originate from initial conditions that differ only in one cell c_i from one another. Consequently, a positive MLE indicates diverging phase space trajectories, whereas converging trajectories are recognized by a negative MLE. CA for which the former holds are commonly referred to as unstable, whereas CA are referred to as stable if their MLE is negative. Clearly, in practice Eq. (1) can only be evaluated for a finite number of time steps, say T. Accordingly, we denote the MLE obtained by considering T time steps as $\lambda_1(T)$, *i.e.*

$$\lambda_1(T) = \frac{1}{T} \log\left(\frac{\varepsilon_T}{\varepsilon_0}\right). \tag{2}$$

It should be emphasized that the numerically obtained MLE can depend on the choice of the initially perturbed cell, and hence on the employed initial condition, which can be accounted for by adding some noise to the CA's evolution [10] or by repeatedly computing the MLE over a series of initial perturbations. An upper bound on the MLE of one-dimensional elementary cellular automaton is given by [10]:

$$\lambda_1 = \log(3\,\bar{\mu}), \tag{3}$$

where $\bar{\mu}$ represents the time- and space-averaged proportion of cells that belong to the neighborhood of a cellular automaton's cell, and which affects the outcome of one consecutive update. Mathematically, this proportion is defined as:

$$\bar{\mu} = \left(\frac{1}{3\,n} \prod_{t=1}^{T} \sum_{c_i} \sum_{j=-1}^{1} J_{i,i+j}\right)^{\frac{1}{T}}, \tag{4}$$

where $J_{i,i+j}$ is one if changing the state of c_{i+j} alters the c_i's at the subsequent time step, and zero otherwise. Clearly, since $\bar{\mu} \in [0,1]$, the MLE of one-dimensional CA is confined between 0 and $\log(3) \approx 1.09861$. Originally, Eqs. (1)-(4) were proposed to grasp the dynamics of elementary CA, but recently, Baetens and De Baets [15] extended its usability to any kind of dimension and tessellation, and further used it as a starting point to formalize the topological sensitivity of CA [22]. For more details on the computation of $\bar{\mu}$ the reader is referred to [10], whereas an algorithm for accurately calculating ε_t can be found in [15].

The full Lyapunov spectrum

The dynamics of an elementary cellular automaton of which the states of cell c_i at the t-th time step are denoted as $s_i(t)$ is governed by

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \\ \cdots \\ s_{n-1}(t+1) \\ s_n(t+1) \end{bmatrix} = \begin{bmatrix} \phi_1\left(\tilde{s}(N(c_i),t)\right) \\ \phi_2\left(\tilde{s}(N(c_i),t)\right) \\ \cdots \\ \phi_{n-1}\left(\tilde{s}(N(c_i),t)\right) \\ \phi_n\left(\tilde{s}(N(c_i),t)\right) \end{bmatrix}, \tag{5}$$

where $\phi_i : S^{|N(c_i)|} \to S$ is a so-called transition function that determines the state of c_i in the subsequent time step based upon the states of c_i's neighboring cells $N(c_i)$ at the t-time step that are given by $\tilde{s}(N(c_i),t)$, and which is applied iteratively for a given number of time steps. At the scale of its individual cells, every c_i has its proper phase space S, but of more interest for studying the dynamics of CA, is that, globally, a cellular automaton may be regarded as a mapping $\Phi : S^n \to S^n$, which follows from the locally applied transition functions f_i. From Eq. (5), it is clear that the phase space S^n of a elementary CA has n dimensions, such that any such CA must have n Lyapunov exponents since the full Lyapunov spectrum of a discrete dynamical system consists of as many Lyapunov exponents as the number of dimensions its phase space is composed of [23].

As such, the MLE given by Eq. (1) merely constitutes the largest Lyapunov exponent among the exponents within the full Lyapunov spectrum of a cellular automaton, which we denote as

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) \ , \tag{6}$$

where, by convention, $\lambda_i \geq \lambda_{i+1}$. Naturally, the question rises how Λ can be determined for a given cellular automaton. In order to address this issue, let us first rewrite Eq. (5) more compactly by introducing a vector notation, *i.e.*

$$\mathbf{s}(t+1) = \Phi(\mathbf{s}(t)) \ . \tag{7}$$

Further, recall the Jacobian matrix J of elementary CA, which can be computed by resorting to the notion of Boolean derivatives as introduced by Vichniac [16]. Essentially, since $N(c_i) = (c_{i-1}, c_i, c_{i+1})$ for elementary CA, J constitutes a tridiagonal matrix of which the entries along the main diagonal are given by

$$\frac{\partial s_i(t+1)}{\partial s_i(t)} = \phi\left(s_{i-1}(t), \bar{s}_i(t), s_{i+1}(t)\right) \oplus \phi\left(s_{i-1}(t), s_i(t), s_{i+1}(t)\right) \ ,$$

where $\bar{s}_i(t)$ denotes the Boolean complement of $s_i(t)$ and, accordingly for the elements along the diagonals below and above this main diagonal. By analogy with its interpretation in the case of dynamical systems that are built upon a continuous state space, J is to be understood as acting on a 'small' Boolean increment, which may be envisaged as a single one in a background of zeros [16]. Hence, J can be employed to retrieve the evolution in tangent space of a perturbation that was introduced during the cellular automaton's evolution, *i.e.*

$$\mathbf{d}(t+1) = J\left(\mathbf{s}(t)\right) \cdots \mathbf{d}(t) \, , \tag{8}$$

where $\mathbf{d}(\mathbf{t})$ denotes a so-called damage vector that contains the number of defects in every c_i at the t-th time step.

Now, let us consder the matrix

$$J_{\mathbf{s}}^t = J\left(\Phi^{t-1}\mathbf{s}\right) \ldots J\left(\Phi\mathbf{s}\right) J\left(\mathbf{s}\right) \ , \tag{9}$$

where, for reasons of brevity, $\Phi^k\mathbf{s}$ is introduced to denote the k-th iterate in the cellular automaton's evolution. Then, again by analogy with dynamical systems that are built upon a continuous state space, we can calculate

$$\lim_{t\to\infty} \left(J^{t*} J^t\right)^{\frac{1}{2t}} = \Xi_{\mathbf{s}} \, , \tag{10}$$

of which the existence is granted by the multiplicative ergodic theorem of Oseledec [24], and which can be obtained straightforwardly using Eq. (9). In Eq. (10) J^{t*} denotes the adjoint of J^t. By definition, the logarithms of the eigenvalues of $\Xi_{\mathbf{s}}$ are the Lyapunov exponents Λ that characterize the dynamics of Φ, and, as such, these logarithms may be envisaged to represent the Lyapunov exponents of a cellular automaton. The computation of the eigenvalues of $\Xi_{\mathbf{s}}$ should be undertaken with care since numerical problems unavoidably arise due to the different orders of magnitude that Λ's elements have, which can be overcome by using an appropriate QR decomposition [25], or, if $J_{\mathbf{s}}^t$'s entries are natural numbers, as is the case for elementary CA, by resorting to computational methods that allow to compute $\Xi_{\mathbf{s}}$'s eigenvalues exactly. In accordance with the meaning put on $\lambda_1(T)$, $\Lambda(T)$ denotes the Lyapunov spectrum obtained by taking into account T times steps of the cellular automaton's evolution.

THE LYAPUNOV SPECTRUM OF ELEMENTARY CELLULAR AUTOMATA

The results presented in this section were obtained for simulations of the 88 minimal elementary CA, as defined in [16], that were based upon a one-dimensional array $\mathscr{T}$ of $n = 100$ cells c_i, which was supplemented with periodic

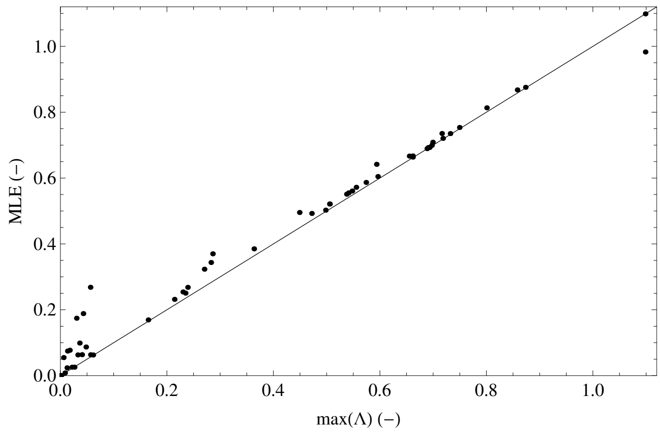

FIGURE 1. Maximum Lyapunov exponent (MLE) obtained numerically by means of Eq. (2) versus the largest exponent in the full Lyapunov spectrum Λ computed by relying on Eq. (10) for those CA within the family of 88 minimal elementary CA that give rise to a real-valued MLE and $\max(\Lambda)$.

boundary conditions to mimic the presupposed infiniteness of the spatial domain. Further, the reported MLEs and Lyapunov spectra were obtained numerically for $T = 500$ since by then $\Lambda(T)$'s entries showed convergence in the sense that $|\lambda_1(T) - \lambda_1(T+1)| \leq 10^{-3}$, and analogously for $\bar{\mu}(T)$ and the exponents in $\Lambda(T)$. For comprehensiveness, it should be emphasized that the reported $\Lambda(T)$ represent averages that were obtained over an ensemble $E_1 = \{{}_e\mathbf{s}(0) \mid e = 1,\ldots,8\}$ of eight different random initial conditions. Accordingly, for every ${}_e\mathbf{s}(0)$ in E_1, the MLE was evaluated numerically using Eq. (1), but moreover, this computation was repeated for each of the n initial perturbations ${}_{e,f}\mathbf{s}^*(0)$, $f = 1,\ldots,n$ that can be envisaged for a given ${}_e\mathbf{s}(0)$ in E_1, *i.e.* over an ensemble of initial perturbations $E_2 = \left\{{}_{e,f}\mathbf{s}^*(0) \mid f = 1,\ldots,n\right\}$, and, as such the reported MLE represent averages over the ensembles E_1 and E_2. In order to overcome numerical problems that typically arise when computing the eigenvalues $\Xi_\mathbf{s}$, which can be attributed to the numerical errors that arise by expressing $L^t_\mathbf{s}$'s elements as doubles, and by relying on the fact that these elements must necessarily be natural numbers if the dynamical system at stake is a cellular automaton, the eigenvalues of $\Xi_\mathbf{s}$ were computed exactly.

Validity of the proposed methodology

In order to verify the validity of the proposed methodology for determining the full Lyapunov spectrum of elementary CA, Figure 1 depicts the numerically obtained MLE (Cfr. Eq. (2)) versus the maximum Lyapunov exponent in Λ (Cfr. Eq. (10)) for those CA within the family of 88 minimal elementary CA that give rise to a real-valued MLE and $\max(\Lambda)$. This figure clearly indicates the good agreement between $\max(\Lambda)$ on the one hand, and the MLE that was obtained numerically using Eq. (1) and Algorithm 1 given in [22], on the other hand, which is confirmed quantitatively by a Pearson correlation coefficient of 0.99. In line with the findings of Bagnoli et al. [10], the highest $\max(\Lambda)$ is observed for rules 105 and 150, and equals the theoretical upper bound on the MLE of elementary CA, given by Eq. (3) in which we put $\bar{\mu} = 1$ since the spread of defects, and hence the divergence rate in phase space, is highest if every cell in $N(c_i)$ is affected by a defect in c_i and this holds for all $c_i \in \mathscr{T}$. Even though there are some rules for which the MLE slightly deviates from $\max(\Lambda)$, which will be studied more closely in future work, the overall good agreement between the MLE and $\max(\Lambda)$ and the agreement between the numerically obtained and theoretically derived spectrum for certain rules (not reported here) suggests that the above outlined procedure can be safely relied upon for determining the full Lyapunov spectra of CA.

Some important Lyapunov spectra

In previous studies it has been shown that the overall behavior of phase space trajectories can be determined by considering the MLE's sign since CA for which $\lambda_1 > 0$ are sensitive to the imposed initial conditions, meaning

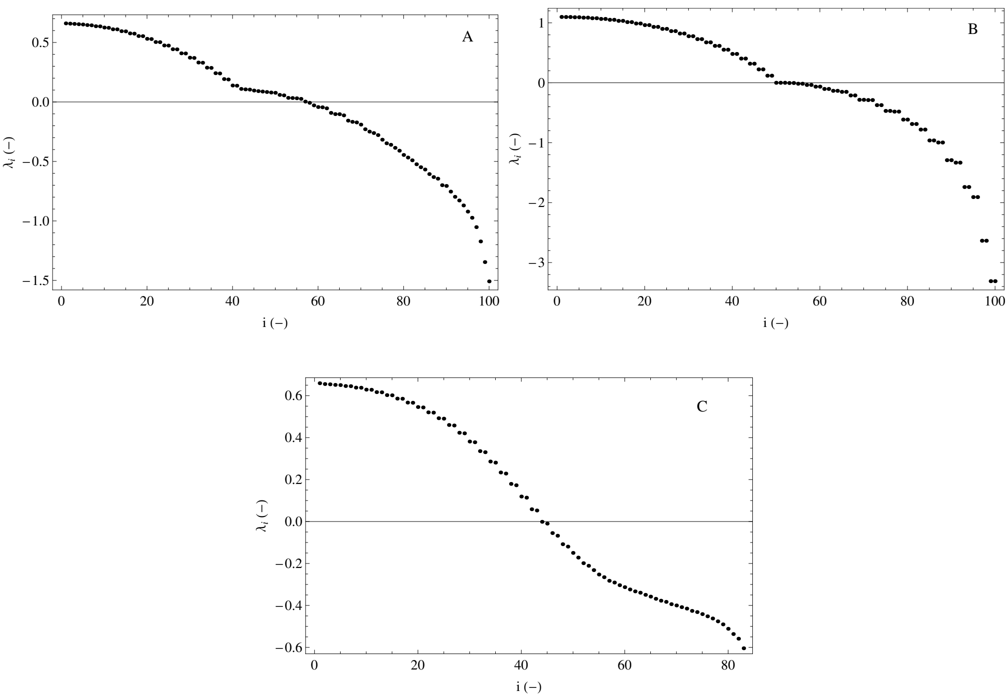

FIGURE 2. Full Lyapunov spectra for rules (A) 30, (B) 105 and (C) 110 showing those elements in Λ for which $\lambda_k > -\infty$.

that the trajectories in phase space originating from close initial conditions $\mathbf{s}(0)$ and $\mathbf{s}^*(0)$ diverge, whereas CA for which $\lambda_1 < 0$ are insensitive to perturbations of $\mathbf{s}(0)$. By means of the full Lyapunov spectrum we can get more profound insights into the specificities of the underlying attractors by counting the number of positive, negative and zero exponents [26] and by relying on the Kaplan-York conjecture [27]. Yet, within the framework of this short paper, we confine the discussion to the Lyapunov spectra of some well-known elementary CA that are of interest in many papers on CA dynamics, namely rules 30, 105 which belong to Wolfram's [9] third behavioral class, and rule 110 that has been categorized under Wolfram's [9] fourth behavioral class.

Figure 2 depicts the full Lyapunov spectra of rules 30, 105 and 110. From this figure it is clear that every spectrum is composed of both negative and positive exponents, whereas zero Lyapunov exponents are only retrieved in the spectrum of rule 105 for which three exponents equal zero. Though each of these rules is sensitive to the imposed initial conditions because $\max(\Lambda) > 0$, and as such may be referred to as chaotic [28], there are some differences between the spectra of both Class 3 rules on the one hand, and rule 110 on the other hand. Indeed, the former spectra clearly show the presence of a kink occurring between the 40th and 50th index, but in contrast to the spectrum of rule 110 which is not endowed with such a kink, the curvature of the graph that connects the successive indices in each of the former spectra does not change. Further, we note that the range of the Lyapunov spectrum is much wider in the case of Class 3 rules as opposed to rules belonging to Class 4. Similar observations were made with regard to other rules belonging to Wolfram's [9] Classes 3 and 4. Therefore, a continued exploration of the Lyapunov spectra of CA might uncover a means to discriminate between Class 3 and Class 4 CA though one should be aware that this might constitute a quest that lacks a clear outcome since the distinction between these classes is motivated by the visual detection of so-called complex, localized structures whereas, from a dynamical systems viewpoint, there might not exist such a distinction. In this respect, it can be understood easily that one attains a clustering of elementary CA which coincides with Wolfram's [9] classification by employing the size of the compressed space-time patterns to characterize the CA dynamics because such a methodology, as proposed in [11], uses exactly the same patterns that are at the basis of Wolfram's [9] typification. Clearly, the evolved space-time patterns do not necessarily embody the characteristics of a CA's phase space trajectories as quantified by means of the MLE.

CONCLUSIONS AND PERSPECTIVES

Motivated by the importance of the full Lyapunov spectra for the characterization of both continuous dynamical systems, coupled-map lattices and maps, we presented the first steps towards a sound characterization of CA by means of their full Lyapunov spectra. More specifically, we outlined the methodology which should be adhered to for determining a cellular automaton's full spectrum, we showed its reliability, and, finally, we presented the full spectra of three elementary CA. On the basis of our preliminary results it may be concluded that the full Lyapunov spectrum of CA can be determined accurately by relying on the notion of Boolean derivatives, and by resorting to methods that allow its exact computation, which takes away the need for time-consuming orthogonalization procedures. Of course, further research should be directed towards the analysis of such spectra and their significance in understanding the intriguing dynamics of CA.

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